Ultra-Relativistic Hamiltonian with Various Singular Potentials

Atsushi Yoshida

Department of Physics, University of Virginia, Charlottesville, VA 22901

(August 24, 1999)

Abstract

It is shown from a simple scaling invariance that the ultra-relativistic Hamiltonian (μ=0) does not have bound states when the potential is Coulombic. This supplements the application of the relativistic virial theorem derived by Lucha and Schöberl [1,2] which shows that bound states do not exist for potentials more singular than the Coulomb potential.

PACS numbers:03.65.Ge, 12.39.Ki, 12.39.Pn

Typeset using REVTEX

The relativistic generalization of the Schrödinger equation (RSE)

$$\sqrt{\mathbf{p}^2+\mu^2}\psi(\mathbf{x})+V(r)\psi(\mathbf{x})=(E+\mu)\psi(\mathbf{x}) \tag{1}$$

has been used in describing quark-antiquark bound states when one of the constituents is light and the other heavy [3]. The mass μ can be considered as the constituent mass of the light quark and E is the binding energy.

This Hamiltonian has two interesting critical behaviors concerning the existence of bound states when the potential is Coulombic,

$$V_c=-\alpha/r, \qquad \alpha>0. \tag{2}$$

First, it has been shown that, irrespective of the value of μ, when the coupling constant of the Coulomb potential α approaches the value $2/\pi$ from below, the bound states disappear due to the large singularity of the potential at the origin [4]. This means, as a corollary, that there are no bound states when the potential is more singular than Coulomb. In this letter, I show the second critical behavior concerning the existence of bound states for a Coulombic potential: irrespective of the value of coupling constant, bound states do not exist for a massless particle ($\mu=0$). This second critical behavior in terms of μ implies (similar to the what is implied by the first critical behavior in terms of α) that there are no bound states with massless particles for a potential more singular than Coulomb.

First I derive a scaling feature of RSE for a general potential. Then I concentrate on the Coulomb potential and show that bound states do not exist for massless particles when the potential is Coulombic. I start with two wavefunctions $\psi(\mathbf{x})$ and $\tilde{\psi}(\mathbf{x})$ and their Fourier transforms $\phi(\mathbf{p})$ and $\tilde{\phi}(\mathbf{p})$ related in the following way,

$$\psi(\mathbf{x})\equiv\tilde{\psi}(t\mathbf{x}) \tag{3}$$

$$\begin{aligned}\phi(\mathbf{p}) &= \frac{1}{(\sqrt{2\pi})^3}\int_{-\infty}^{\infty}\psi(\mathbf{x})e^{i\mathbf{p}\cdot\mathbf{x}}dx^3\\ &= \frac{1}{t^3}\frac{1}{(\sqrt{2\pi})^3}\int_{-\infty}^{\infty}\tilde{\psi}(t\mathbf{x})e^{i\frac{\mathbf{p}}{t}\cdot(t\mathbf{x})}d(tx)^3\\ &= \frac{1}{t^3}\tilde{\phi}(\frac{\mathbf{p}}{t}).\end{aligned} \tag{4}$$

Here t can be any positive real number. Now suppose $\psi(\mathbf{x})$ satisfies RSE, then

$$
\begin{aligned}
(E+\mu)\psi(\mathbf{x}) &= \sqrt{\mathbf{p}^2+\mu^2}\psi(\mathbf{x}) + V(\mathbf{x})\psi(\mathbf{x}) \\
&= \frac{1}{(\sqrt{2\pi})^3}\int dp^3 \sqrt{\mathbf{p}^2+\mu^2}\phi(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} + V(\mathbf{x})\psi(\mathbf{x}) \\
&= \frac{1}{t^3}t^4\frac{1}{(\sqrt{2\pi})^3}\int d\Big(\frac{p}{t}\Big)^3 \sqrt{\Big(\frac{\mathbf{p}}{t}\Big)^2+\Big(\frac{\mu}{t}\Big)^2}\tilde{\phi}(\frac{\mathbf{p}}{t})e^{-i\frac{\mathbf{p}}{t}\cdot(t\mathbf{x})} + V(\mathbf{x})\tilde{\psi}(t\mathbf{x}) \\
&= t\sqrt{\mathbf{p}^2+\Big(\frac{\mu}{t}\Big)^2}\tilde{\psi}(t\mathbf{x}) + V(\mathbf{x})\tilde{\psi}(t\mathbf{x}) \qquad (5)
\end{aligned}
$$

Thus, using the equivalence (3), it gives

$$\sqrt{\mathbf{p}^2+\Big(\frac{\mu}{t}\Big)^2}\tilde{\psi}(\mathbf{x}) + \frac{V(\frac{\mathbf{x}}{t})}{t}\tilde{\psi}(\mathbf{x}) = \Big(\frac{E}{t}+\frac{\mu}{t}\Big)\tilde{\psi}(\mathbf{x}). \qquad (6)$$

If the potential is Coulomb $V(\mathbf{x}) = -\alpha/r$, then we arrive at the scaling results that if

$$\sqrt{\mathbf{p}^2+\mu^2}\psi(\mathbf{x}) - \frac{\alpha}{r}\psi(\mathbf{x}) = (E+\mu)\psi(\mathbf{x}) \qquad (7)$$

then

$$\sqrt{\mathbf{p}^2+\tilde{\mu}^2}\tilde{\psi}(\mathbf{x}) - \frac{\alpha}{r}\tilde{\psi}(\mathbf{x}) = (\tilde{E}+\tilde{\mu})\tilde{\psi}(\mathbf{x}), \qquad (8)$$

with the relations

$$
\begin{aligned}
\tilde{\psi}(\mathbf{x}) &= \psi(\frac{\mathbf{x}}{t}) && (9)\\
\tilde{\mu} &= \frac{\mu}{t} && (10)\\
\tilde{E} &= \frac{E}{t}. && (11)
\end{aligned}
$$

In fact this theorem applies to slightly more general potentials. As long as the parameters of the potential $V(\mathbf{x})$ are all dimensionless, then

$$\frac{1}{t}V(\frac{\mathbf{x}}{t}) = V(\mathbf{x}). \qquad (12)$$

This is the only criteria that the potential must satisfy. In particular, the potential need not be spherically symmetric. For example potentials like

$$V(\mathbf{x}) = -\frac{\alpha_1}{\sqrt{x^2+y^2}} - \frac{\alpha_2}{|z|} \qquad \text{(cylindrical)} \qquad (13)$$

and

$$V(\mathbf{x}) = -\frac{\alpha}{\sqrt{ax^2+by^2+cz^2}} \qquad \text{(ellipsoidal)} \tag{14}$$

do satisfy eq.(12). The underlying reason for this generality is that if the potential has only dimensionless parameters, the wavefunction is restricted to the form

$$\psi(\mathbf{x}) = \mu^{\frac{3}{2}} f(\mu\mathbf{x}; \{\alpha_i\}). \tag{15}$$

Then after normalization, eq.(9) reduces to the statement

$$\text{if} \quad \psi(\mathbf{x}) = \mu^{\frac{3}{2}} f(\mu\mathbf{x}; \{\alpha_i\}) \quad \text{then} \quad \tilde{\psi}(\mathbf{x}) = \tilde{\mu}^{\frac{3}{2}} f(\tilde{\mu}\mathbf{x}; \{\alpha_i\}). \tag{16}$$

Here the normalization is $\int dx^3 |f(\mathbf{x}; \{\alpha_i\})|^2 = 1$.

In fact, from a dimensional point of view alone, one can show that eq.(15) must hold for non-relativistic Schrödinger equation (NRSE), and from this, all the relations (9),(10) and (11) as well.

This scaling behavior indicates, in words, that as the mass increases by a factor of t, the wavefunction shrinks by a factor of t and the binding energy is also increased by a factor of t (in magnitude). From this, it is clear that for $\tilde{\mu} = 0$, $\mu = 0$ automatically as well. This means now the Hamiltonians (7) and (8) become identical. Since this is true for any arbitrary t, it follows $\tilde{E} \to -\infty$ as $t \to 0$. Hence there are no massless bound states.

One can also extract simple scaling laws for other types of potentials from eq.(6). In particular, for massless case with a potential $V(r) = -r^{-k}$ with $k > 1$, one can again show that bound states do not exist by taking the limit $t \to 0$. This time the potential is not invariant but

$$\tilde{V}(\frac{r}{t}) = t^k V(r). \tag{17}$$

Therefore, as $t \to 0$ the potential becomes shallower, nevertheless, still the wavefunction shrinks to zero width and the energy goes to negative infinity. Thus again there are no bound states. This is reminiscent to the well-known theorem in NRSE which states that for a potential more singular than r^{-2} bound states do not exist [5]. Here in our situation, since massless bound states do not exist for a Coulomb potential it is intuitively clear that for a potential more singular, the same is true.

This fact that the bound states do not exist for a potential more singular than the Coulomb potential can be inferred also from the relativistic virial theorem (RVT) derived by Lucha and Schöberl [1,2]. It states, for an eigenstate of two-body relativistic Hamiltonian (in the center-of-mass frame)

$$H = \sqrt{\mathbf{p}^2 + m_1^2} + \sqrt{\mathbf{p}^2 + m_2^2} + V(x), \tag{18}$$

the gradient of the potential is related to the kinetic energy as

$$\langle \mathbf{x} \cdot \nabla V(\mathbf{x}) \rangle = \langle \frac{\mathbf{p}^2}{\sqrt{\mathbf{p}^2 + m_1^2}} + \frac{\mathbf{p}^2}{\sqrt{\mathbf{p}^2 + m_2^2}} \rangle. \tag{19}$$

This leads to

$$\varepsilon = \langle \mathbf{x} \cdot \nabla V(\mathbf{x}) \rangle + \langle V(\mathbf{x}) \rangle + \langle \frac{m_1^2}{\sqrt{\mathbf{p}^2 + m_1^2}} + \frac{m_2^2}{\sqrt{\mathbf{p}^2 + m_2^2}} \rangle \tag{20}$$

where ε denotes the total energy of the two-body system of particle mass m_1 and m_2. The Hamiltonian (18) simplifies to (1) when m_2 is taken to infinity and m_1 is set to be μ. Accordingly (20) also reduces to

$$E = \langle \mathbf{x} \cdot \nabla V(\mathbf{x}) \rangle + \langle V(\mathbf{x}) \rangle + \langle \frac{\mu^2}{\sqrt{\mathbf{p}^2 + \mu^2}} \rangle. \tag{21}$$

Here the lhs has been reduced to the binding energy E. For a radially symmetric power law potential

$$V(r) = \alpha r^k \tag{22}$$

where α is positive for $k > 0$ and negative for $k < 0$, this means simply

$$E = (k+1)\langle V \rangle + \langle \frac{\mu^2}{\sqrt{\mathbf{p}^2 + \mu^2}} \rangle. \tag{23}$$

Clearly for $k < -1$, if bound states were to exist, it would give a nonsensical result because the lhs is negative while rhs is positive. This indicates that the eigenstates themselves do not exist for those potentials with $k < -1$ for both finite and zero μ. Literally taken, RVT predicts that all the massless particle bound states have one and the same binding energy $E = 0$ for a Coulomb potential ($k = -1$). This could also be seen as a manifestation that the bound states do not exist and the expectation values can not be defined.

Incidentally, in the non-relativistic case, not only these scaling relations eqs.(9), (10), (11) and (17) hold, but more general scaling relations can be derived. Indeed, NRSE with a potential of the form $V(r) = \alpha r^k$, where α is positive when $k > 0$, and negative when $k < 0$, can be converted in radially reduced form to the following dimensionless equation [6],

$$-\frac{d^2}{d\rho^2} w(\rho) + [sgn(\alpha)\rho^k + \frac{l(l+1)}{\rho^2}] w(\rho) = \epsilon w(\rho). \tag{24}$$

From this, it can be seen that all the scaling relations derived here are only a part of this more general scaling transformation. Unfortunately, for the RSE case, this general transformation to the dimensionless form, which would allow one to extract a lot more informations on the bound states, does not seem to be possible.

I thank D.Singleton for reading the manuscript.

REFERENCES

[1] W.Lucha and F.F.Schröberl, Phys. Rev. Lett. **64** 2733 (1990)

[2] W.Lucha and F.F.Schröberl, Mod. Phys. Lett. **A5** 2473 (1990)

[3] A.Duncan, E.Eichten and H.Thacker, Nucl. Phys. **B**(Proc. Suppl.)**30**, 441 (1993)

[4] I.W.Herbst, Comm. Math. Phys. **53** 285 (1977); **55** 316 (1977) (addendum)

[5] L.D.Landau and E.M.Lifshitz *Quantum Mechanics*, (Pergamon Press, New York (1987))

[6] C.Quigg and J.L.Rosner, Phys. Rep. **56**, 167 (1979)